

The matrix Lax representation of the generalized Riemann equations and its conservation laws

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Abstract

It is shown that the generalized Riemann equation is equivalent with the multicomponent generalization of the Hunter - Saxton equation. New matrix and scalar Lax representation is presented for this generalization. New class of the conserved densities, which depends explicitly on the time are obtained directly from the Lax operator. The algorithm, which allows us to generate a big class of the non-polynomial conservation laws of the generalized Riemann equation is presented. Due to this new series of conservation laws of the Hunter-Saxton equation is obtained.

Introduction.

The theory of the hydrodynamical type systems of the non-linear equations [1] , integrable by the generalized hodograph method [2] is closely related to the overdetermined systems of first order partial differential linear equations . It is achieved introducing the so called Riemann invariants in which the hydrodynamic type system is rewritten in the diagonal form as $r_t^i = \mu(r)^i r_x^i$ where $i = 1, 2 \dots, r = (r_1, r_2 \dots r_N)$ and no summation on the repeated indices. Thus, it is a linear systems of first order partial differential equations with variable coefficients $\mu(r)$.

When $N = 1$ the equation on Riemann invariant reduces to the so called Riemann equation $r_t = -rr_x$ which have been investigated in many papers and could be considered as the dispersionless limit of the Korteweg de Vries equation [3]. Recently the interesting generalization of the Riemann equation to the multicomponent case $(\partial/\partial t + u\partial/\partial x)^N u = 0, N = 1, 2 \dots$ have been proposed in [4, 5, 6]

When $N = 2$ this generalized system is reduced to the Gurevich-Zybin system [7, 8] or to the equation which describes the non-local gas dynamic [9] or to the Whitham type system [1]. It is possible also to reduce the $N = 2$ generalized Riemann equation [4] to the celebrated Hunter-Saxton equation [10], sometimes referred as the Hunter-Zheng equation [11]. The Hunter-Saxton equation has been

studied in almost all respects, including its complete solvability by quadratures [12, 13], construction of an infinite number of conservation laws [10, 11, 14], relationship with the Camassa-Holm equation and the Liouville equation [15], Bi-Hamiltonian formulation [11, 15], integrable finite-dimensional reductions [11, 16], global solution properties [17, 18], to mention only a few of numerous publications on this equation.

For an arbitrary N the investigation of the properties of the generalized Riemann equation just started in [4, 5, 6]. It was indicated that $N = 3$ generalized Riemann equation possess the matrix Lax representation, the Hamiltonian formulations and huge number of polynomial and non-polynomial conservation laws. However this matrix Lax representation is a free-form, because it contains one arbitrary function which could be fixed but in not a unique manner, taking into account the integrability condition on the Lax representation.

In this paper we present the matrix Lax representation for an arbitrary N which is not a free-form. This representation for $N = 2$ could be reduced to the very well known energy-dependent second-order Lax operator [19, 20, 13] while for $N = 3$ to the energy depended third-order Lax operator introduced in [21]. Some of the functions, which constitute the scalar Lax pair for $N = 2, 3$, are also the non-polynomial conserved Hamiltonian functionals. Moreover from this matrix Lax representation it is possible to obtain the conserved densities which are explicitly time depended. We present the operator, in some sense the analogue of the recursion operator, which generates an infinite number of non-polynomial conservation laws. As the by-product of our analysis we present new series of the non-polynomial conservation laws for the Hunter-Saxton equation.

The paper is organised as follows. The first section describes the generalized Riemann equation and shows its connection with the multi-component generalizations of the Hunter-Saxton equation. In the second section we define new matrix representation for an arbitrary N extended Riemann equation. The third section describes the reduction of matrix Lax representation, for $N = 2$ and 3 , to the scalar Lax representation. In the fourth section the conservation laws for $N = 2, 3$ generalized Riemann equation are obtained directly from the Lax representation. The fifth section describes the algorithm of generations of the non-polynomial conservation laws.

1 Generalized Riemann Equation.

The hydrodynamical Riemann equation

$$u_t = -uu_x, \quad (1)$$

have been recently generalized to the multicomponent case [4, 5, 6] as

$$\mathcal{D}_t^N u = 0, \quad \mathcal{D}_t = \partial/\partial t + u\partial/\partial x, \quad (2)$$

where $N = 1, 2, 3, \dots$. We can rewrite the last equation to the more comfortable form, introducing the notation $u_1 = u, u_n = \mathcal{D}^{n-1}u$ and then the N generalized

Riemann equation takes the following form

$$\begin{aligned}
u_{1,t} &= u_2 - u_1 u_{1,x}, \\
u_{2,t} &= u_3 - u_1 u_{2,x}, \\
&\dots\dots\dots \\
u_{N-1,t} &= u_N - u_1 u_{N-1,x}, \\
u_{N,t} &= -u_1 u_{N,x}.
\end{aligned} \tag{3}$$

The multicomponent generalization of the Hunter-Saxton equation could be obtained from the last formula using the transformation $u_{2,x} = (u_{1,x}^2 + w_2^2)/2$, $u_{k,x} = w_2^2 w_k$ for $k = 3, 4, \dots$

$$\begin{aligned}
u_{1,t,x} &= -\frac{u_{1,x}^2}{2} - u_1 u_{1,xx} + \frac{w_2^2}{2}, \\
w_{2,t} &= -(u_1 w_2)_x + w_3 w_2, \\
w_{3,t} &= u_{1,x} w_3 - u_1 w_{3,x} - 2w_3^2 + w_4, \\
&\dots\dots\dots \\
w_{N-1,t} &= u_{1,x} w_N - u_1 w_{N-1,x} - 2w_3 w_{N-1} + w_N, \\
w_{N,t} &= u_{1,x} w_N - u_1 w_{N,x} - 2w_3 w_N.
\end{aligned} \tag{4}$$

When all w_i vanishes then Esq. (4) reduces to the Hunter-Saxton equation while for $w_i = 0, i = 3, 4, \dots$ our equations reduce to the two-component Hunter-Saxton equation

2 Matrix Lax representation.

The generalized Riemann equation for $N > 1$ could be obtained from the compatibility condition for the matrix Lax representation

$$\Psi_x = A\Psi, \quad \Psi_t = -u_1 A\Psi - \lambda E\Psi, \tag{5}$$

where $\Psi = (\psi_1, \psi_2, \dots, \psi_N)^T$, A, E are $N \times N$ matrices such that $E_{k,k-1} = 1$ for $k = 1, 2, \dots, N-1$ and

$$A = \begin{pmatrix} \lambda^2 u_{N-1,x} & \lambda u_{N,x} & 0 & \dots & 0 \\ 0 & \lambda^2 u_{N-1,x} & 2\lambda u_{N,x} & \dots & 0 \\ & \vdots & \vdots & \vdots & \\ \dots & \dots & \lambda^2 u_{N-1,x} & k\lambda u_{N,x} & \dots \\ & \vdots & \vdots & \vdots & \\ -N\lambda^{N+1} & -N\lambda^N u_{1,x} & \dots & -N\lambda^3 u_{N-2,x} & -(N-1)\lambda^2 u_{N-1,x} \end{pmatrix} \tag{6}$$

Explicitly for $N = 2, 3$ we obtain

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \lambda^2 u_{1,x} & \lambda u_{2,x} \\ -2\lambda^3 & -\lambda^2 u_{1,x} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{7}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = \begin{pmatrix} -\lambda^2 u_1 u_{1,x} & -\lambda u_1 u_{2,x} \\ \lambda(2\lambda^2 u_1 - 1) & \lambda^2 u_1 u_{1,x} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{8}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_x = \begin{pmatrix} \lambda^2 u_{2,x} & \lambda u_{3,x} & 0 \\ 0 & \lambda^2 u_{2,x} & 2\lambda u_{3,x} \\ -3\lambda^4 & -3\lambda^3 u_{1,x} & -2\lambda^2 u_{2,x} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (9)$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}_t = \begin{pmatrix} -\lambda^2 u_1 u_{2,x} & -\lambda u_1 u_{3,x} & 0 \\ -\lambda & -\lambda^2 u_1 u_{2,x} & -2u_1 \lambda u_{3,x} \\ 3\lambda^4 u_1 & 3\lambda^3 u_1 u_{1,x} - \lambda & 2\lambda^2 u_1 u_{2,x} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \quad (10)$$

3 Scalar Lax representation for N=2,3.

We consider the case $N = 2$ and $N = 3$ separately.

3.1 N=2.

We can obtain two different scalar Lax representation for $N = 2$ generalized Riemann equation .

For the first choice computing Ψ_1 from the second equation in the formula 7 we obtain

$$\begin{aligned} \Psi_{2,xx} &= \lambda^2(-2\lambda^2 u_{2,x} - u_{1,xx} + \lambda^2 u_{1,x}^2) \Psi, \\ \Psi_{2,t} &= \left(\frac{1}{2\lambda^2} - u_1\right) \Psi_{2,x} + \frac{1}{2} u_{1,x} \Psi_2. \end{aligned} \quad (11)$$

It is exactly the Lax operator considered in [20, 13].

For the second case computing ψ_2 from the first equation in 7 and using the transformation $\psi_1 \Rightarrow \sqrt{u_2} \Phi$ we obtain

$$\begin{aligned} \Phi_{xx} &= (\lambda^4 Z_2 + \lambda^2 Z_1 + Z_0) \Phi, \\ \Phi_t &= (u_{1,x} \Phi - 2u_1 \Phi_x) / 2. \end{aligned} \quad (12)$$

where

$$Z_2 = -2u_{2,x} + u_{1,x}^2, \quad Z_1 = u_{2,x} \left(\frac{u_{1,x}}{u_{2,x}}\right)_x, \quad Z_0 = -\frac{1}{4} \frac{2u_{2,xxx} u_{2,x} - 3u_{2,xx}^2}{u_{2,x}^2}. \quad (13)$$

The integrability condition $\Psi_{xx,t} = \Psi_{t,xx}$ gives us

$$\begin{aligned} Z_{i,t} &= -(\partial Z_i + Z_i \partial) u_1, \quad \text{for } i = 1, 2, \\ Z_{0,t} &= \left(\frac{1}{4} \partial^3 - 2\partial Z_0 - 2Z_0 \partial\right) u_1. \end{aligned} \quad (14)$$

and it is identically satisfied if we use the definition of Z_2, Z_1, Z_0 .

3.2 N = 3.

Computing ψ_2 and ψ_3 from the first and second equation in 9 respectively and using the transformation $\psi_1 \Rightarrow u_{3,x} \Omega$ we obtain

$$\begin{aligned} \Omega_{xxx} &= \sum_{k=0}^2 \lambda^{2k} S_k \Omega_x + \left(\sum_{k=0}^3 \lambda^{2k} P_k + \frac{1}{2} \sum_{k=0}^2 \lambda^{2k} S_{k,x}\right) \Omega, \\ \Omega_t &= u_{1,x} \Omega - u_1 \Omega_x. \end{aligned} \quad (15)$$

where

$$\begin{aligned}
P_0 &= 0, & P_3 &= 2(3u_{1,x}u_{3,x}u_{2,x} - 3u_{3,x}^2 - u_{2,x}^3), \\
P_2 &= 3\frac{u_{3,xx}u_{2,x}^2}{u_{3,x}} - 3(u_{1,x}u_{3,xx} - u_{3,x}u_{1,xx} + u_{2,x}u_{2,xx}), \\
P_1 &= -\frac{1}{2}\left(\frac{3u_{3,xx}^2u_{2,x}}{u_{3,x}^2} - \frac{1}{u_{3,x}}(u_{3,xxx}u_{2,x} + 3u_{2,xx}u_{3,xx}) + u_{2,xxx}\right), \\
S_2 &= 3(u_{2,x}^2 - 2u_{1,x}u_{3,x}), & S_1 &= 3\frac{1}{u_{3,x}}(u_{2,xx}u_{3,x} - u_{3,xx}u_{2,x}), \\
S_0 &= \frac{1}{u_{3,x}^2}(3u_{3,xx}^2 - 2u_{3,xxx}u_{3,x}).
\end{aligned} \tag{16}$$

This is the energy-dependent third-order Lax operator considered in [21].

From the integrability condition we obtained the following equations of motion on the functions S_i, P_i

$$\begin{aligned}
P_{i,t} &= -(\partial P_i + 2P_i\partial)u_1, \quad \text{for } i = 1, 2, 3, \\
S_{i,t} &= -(\partial S_i + S_i\partial)u_1, \quad \text{for } i = 2, 1, \\
S_{0,t} &= (\partial^3 - \partial S_0 - S_0\partial)u_1.
\end{aligned} \tag{17}$$

For the second choice where we eliminate ψ_3 and ψ_1 from the second and third equation in 9 respectively and using the transformation $\psi_2 \Rightarrow u_{3,x}^{\frac{2}{3}}\Omega$ we obtain

$$\begin{aligned}
\Omega_{xxx} &= \sum_{k=0}^2 \lambda^{2k} \hat{S}_k \Omega_x + \sum_{k=0}^3 \lambda^{2k} \hat{P}_k \Omega, \\
\Omega_t &= \lambda^{-4} W_0 \Omega_{xx} + \sum_{k=0}^2 \lambda^{-2(2-k)} W_{k+1} \Omega_x + \sum_{k=0}^2 \lambda^{-2(2-k)} W_{k+4} \Omega.
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\hat{S}_0 &= \frac{1}{u_{3,x}}(4u_{3,xx}^2 - 3u_{3,xxx}u_{3,x}), \\
\hat{S}_1 &= \frac{-1}{u_{3,x}}u_{3,xx}u_{2,x}, & \hat{S}_2 &= 3u_{2,x}^2 - 6u_{3,x}u_{1,x}
\end{aligned} \tag{19}$$

$$\begin{aligned}
\hat{P}_0 &= \frac{1}{3u_{3,x}^2}(72u_{3,xxx}u_{3,xx}u_{3,x} - 18u_{3,4x}u_{3,x}^2 - 56u_{3,xx}^3), \\
\hat{P}_1 &= \frac{1}{3u_{3,x}^2}(4u_{2,x}u_{3,xx}^2 + 3u_{3,x}^2u_{2,xxx} - 3u_{3,x}(u_{3,xxx}u_{2,x} + 2u_{3,xx}u_{2,xx})), \\
\hat{P}_2 &= \frac{1}{3u_{3,x}}(u_{2,x}(-3u_{3,xxx}u_{3,x} + 4u_{3,xx}^2) - u_{2,xx}(6u_{3,xx}u_{3,x} + 3u_{3,x}^2)), \\
\hat{P}_3 &= (u_{3,xx}(-4u_{3,x}u_{1,x} + u_{2,x}^2) - 3u_{3,x}(2u_{3,x}u_{1,xx} - u_{2,xx}u_{2,x})).
\end{aligned} \tag{20}$$

$$\begin{aligned}
W_0 &= \frac{1}{6u_{3,x}}, & W_1 &= \frac{u_{3,xx}}{18u_{3,x}^2}, & W_2 &= u_{2,x}W_0, & W_3 &= -u_1, \\
W_4 &= \frac{1}{27u_{3,x}^3}(3u_{3,xxx}u_{3,x} - 4u_{3,xx}^2), & W_5 &= \frac{1}{18u_{3,x}^2}(5u_{3,xx}u_{2,x} - 3u_{3,x}u_{2,xx}), \\
W_6 &= \frac{1}{3u_{3,x}}(5u_{3,x}u_{1,x} - u_{2,x}^2).
\end{aligned} \tag{21}$$

In that way Esq. 18 could be considered as the energy-dependent third-order Lax operator also.

4 The conservation laws obtained from Lax representation.

The knowledge of scalar Lax representation allows us to compute the conservation laws for the model.

For the $N = 2$ generalized Riemann equation let us rewrite the first Lax pair in terms of two Riccati equations introducing the function $\Gamma = \Psi_{2,x}/\Psi_2$

$$\begin{aligned}
\Gamma_x &= -\Gamma^2 - \lambda^2 u_{1,xx} - \lambda^4 (2u_{2,x} - u_{1,x}^2), \\
\Gamma_t &= (u_1 - \frac{1}{2\lambda^2})\Gamma^2 - u_{1,x}\Gamma + \lambda^2 (\frac{1}{2}u_{1,x}^2 + u_{1,xx}u_1 - u_{2,x}) + \lambda^4 u_1(2u_{2,x} - u_{1,x}^2).
\end{aligned} \tag{22}$$

Expanding Γ in powers of λ as

$$\Gamma = -\lambda^2 u_{1,x} - 2\lambda^4 u_2 + \sum_{k=0}^{\infty} \lambda^{2k+6} \Gamma_{2k}. \tag{23}$$

we obtain for example the first two integrable equations

$$\begin{aligned}
\Gamma_{0,t} &= 4u_{1,x}u_2u_1 - 2u_2^2, & \Gamma_{2,t} &= -2(u_1u_{1,x} - u_2)\Gamma_0 + 4u_1u_2^2, \\
\Gamma_{0,x} &= -4u_2u_{1,x}, & \Gamma_{2,x} &= 2u_{1,x}\Gamma_0 - 4u_2^2.
\end{aligned} \tag{24}$$

From this follows that all $\Gamma_{n,x}$ are conserved. Indeed

$$\begin{aligned}
H_0 &= \int dx \Gamma_{0,x} = -4 \int dx u_{1,x}u_2, \\
H_2 &= -4 \int dx 2u_{1,x}\partial^{-1}u_{1,x}u_2 + u_2^2 = 4 \int dx u_1^2u_{2,x} - u_2^2.
\end{aligned} \tag{25}$$

These conserved Hamiltonian functionals coincides with those obtained in [9].

Quite different series of conserved densities we obtain using the matrix Lax representation Esq. 9 and 10 in which we redefine the functions Ψ_1 and Ψ_2 as

$$\Psi_1 = e^{(g+\lambda^2 u_1)}, \quad \Psi_2 = \Upsilon e^{(g+\lambda^2 u_1)}. \tag{26}$$

Then our Lax representation gives us

$$\begin{aligned}
g_x &= \lambda u_{2,x}\Upsilon, & g_t &= -u_1g_x - \lambda^2 u_2, \\
\Upsilon_x &= -\lambda u_{2,x}\Upsilon^2 - 2\lambda^2 u_{1,x}\Upsilon - 2\lambda^3, & \Upsilon_t &= -u_1\Upsilon_x - \lambda.
\end{aligned} \tag{27}$$

The integrability conditions $g_{x,t} = g_{t,x}$, $\mathcal{R}_{x,t} = \mathcal{R}_{t,x}$ lead us the $N = 2$ generalized Riemann equation. These two equations are in the conservative form and thus g_x and \mathcal{R}_x are conserved Hamiltonian functionals. The explicit form of these functionals could be obtained expanding the function \mathcal{R} as

$$\mathcal{R} = \frac{1}{u_2^2} \sum_{k=0}^{\infty} \lambda^k \varrho_k, \quad (28)$$

where for example

$$\begin{aligned} \varrho_0 &= 1, & \varrho_1 &= u_2 - t, & \varrho_2 &= 2u_2t + u_2^2 - 2u_1, \\ \varrho_3 &= -t^2u_2 + t(2u_1 - 3u_2^2) - 2x + 2u_1u_2 - u_2^3 + 2\partial^{-1}u_{2,x}u_1. \end{aligned} \quad (29)$$

The first nontrivial conserved Hamiltonian functionals are

$$\begin{aligned} H_1 &= \int dx u_{1,x}u_2^2, & H_2 &= \int dx u_{1,x}u_2^3, \\ H_3 &= \int dx (u_{2,x}u_1^3 - 3u_2^2u_1 - 6u_{2,x}u_1\partial^{-1}u_2). \end{aligned} \quad (30)$$

and explicit time dependent

$$\begin{aligned} H_4 &= \int dx (u_2 - u_{1,x}u_2t), & H_5 &= \int dx (2u_1 - 2u_2t + u_{1,x}u_2t^2) \\ H_6 &= \int dx (u_2^2 + 2u_{2,x}u_2u_1t), & H_7 &= \int dx (u_{2,x}u_1^2 + u_{1,x}u_2^2t), \\ H_8 &= \int dx (-2u_1^2 - 4\partial^{-1}u_2 + 4t\partial^{-1}u_2u_{1,x} - t^2(u_{2,x}u_1^2 + u_2^2)), \end{aligned} \quad (31)$$

while x and time dependent

$$\begin{aligned} H_9 &= \int dx t(6u_2^2u_1 + 12u_{2,x}u_1\partial^{-1}u_2 - 2u_{2,x}u_1^3) + 6x(u_{2,x}u_1^2 + u_2^2), \\ &\quad 12u_1\partial^{-1}u_{2,x}u_1 - 6u_2u_1^2. \end{aligned} \quad (32)$$

Now let us consider once more the equation 11 in which we redefine the functions $\Psi_i, i = 1, 2, 3$ as

$$\Psi_1 = e^{(g+\lambda^2u_2)}, \quad \Psi_2 = \Upsilon e^{(g+\lambda^2u_2)}, \quad \Psi_3 = \Xi e^{(g+\lambda^2u_2)}. \quad (33)$$

and as the result we obtain, after eliminations of Ξ

$$\begin{aligned} g_x &= \lambda u_{3,x}\Upsilon, & g_t &= -u_1g_x - \lambda^2u_3, \\ \Upsilon_{xx} &= \frac{u_{2,xx}}{u_{2,x}}\Upsilon_x - 3\lambda u_{3,x}\Upsilon\Upsilon_x - 3\lambda^2u_{2,x}\Upsilon_x - \lambda^2u_{3,x}^2\Upsilon^3, \\ &\quad -3\lambda^3u_{3,x}u_{2,x}\Upsilon^2 - 6\lambda^4u_{3,x}u_{1,x}\Upsilon - 6\lambda^5u_{3,x} \\ \Upsilon_t &= -u_1\Upsilon_x - \lambda. \end{aligned} \quad (34)$$

The integrability condition $g_{x,t} = g_{t,x}$ and $\Upsilon_{xx,t} = \Upsilon_{t,xx}$ gives us the $N = 3$ generalized Riemann equation. From this representation follows that the g_x generates

conserved Hamiltonian functionals. Indeed if we expand Υ in the power series in λ as

$$\Upsilon = \sum_{k=0}^{\infty} \lambda^k \phi_k, \quad (35)$$

where for example

$$\begin{aligned} \phi_0 &= u_3, & \phi_1 &= -\frac{1}{2}u_3^3 - t, & \phi_2 &= \frac{3}{2}tu_3^2 + \frac{1}{4}u_3^5 - 3\partial^{-1}u_{3,x}u_2, \\ \phi_3 &= -\frac{5}{4}tu_3^4 - \frac{1}{8}u_3^7 - \frac{3}{2}u_3^3u_2 + \frac{9}{2}u_3^2\partial^{-1}u_{3,x}u_2 + \frac{3}{2}u_3\partial^{-1}u_{2,x}u_3^2. \end{aligned} \quad (36)$$

then the coefficients standing in the same power of λ in g_x are conserved Hamiltonian functionals as for example

$$\begin{aligned} H_1 &= \int dx u_{2,x}u_3^2, & H_2 &= \int dx u_3^2(2u_{1,x}u_3 - 3u_{2,x}u_2), \\ H_3 &= \int dx u_3^4(4u_{1,x}u_3 - 5u_{2,x}u_2), \\ H_4 &= \int dx 5u_3^4 + 12u_{2,x}u_3^2\partial^{-1}u_{3,x}u_1 - 12u_{1,x}u_3^2\partial^{-1}u_{3,x}u_2, \\ H_5 &= \int dx 10u_{2,x}u_3^3u_1 + 5u_{3,x}u_3u_2^3 + 6u_3^2(u_{1,x}\partial^{-1}u_{3,x}u_2 - u_{2,x}\partial^{-1}u_{3,x}u_1). \end{aligned} \quad (37)$$

and explicit time dependent

$$\begin{aligned} H_6 &= \int dx u_3^2(-u_{2,x}t^2 + 2u_{1,x}t - 2), & H_7 &= \int dx u_3^6(6u_{3,x}u_2t + u_{2,x}u_2) \\ H_8 &= \int dx 10t^2u_{3,x}u_3^3u_2 + 5tu_{1,x}u_3^4 + 12u_3^2(u_{2,x}\partial^{-1}u_{3,x}u_1 - u_{1,x}\partial^{-1}u_{3,x}u_2), \\ H_9 &= \int dx 2t^2u_{3,x}u_3^3u_2 - 2tu_{3,x}u_3^2u_2^2 - u_{2,x}u_3^2u_2. \end{aligned} \quad (38)$$

5 Non-polynomial conservation laws.

Notice that the equation on $Z_{1,t}$ in (14) can be rewritten as

$$(\sqrt{Z_1})_t = -(u_1\sqrt{Z_1})_x. \quad (39)$$

Similarly the equations on $P_{i,t}$ and on $S_{2,t}, S_{1,t}$ can be rewritten as

$$\begin{aligned} (P_i^{1/3})_t &= -(u_1P_i^{1/3})_x \quad i = 1, 2, 3, \\ (\sqrt{S_i})_t &= -(u_1\sqrt{S_i})_x \quad i = 2, 1. \end{aligned} \quad (40)$$

Therefore $\int dx \sqrt{Z_1}$ and $\int dx \sqrt{S_i}, \int dx P_i^{1/3}$ are conserved Hamiltonian functionals for the $N = 2, 3$ generalized Riemann equation respectively.

Moreover notice that the following functions

$$H_{n,m} = \int dx \left(\frac{(k_1 P_1 + k_2 P_2 + k_3 P_3)^m}{(k_4 S_1 + k_5 S_2)^n} \right)^{\frac{1}{3m-2n}}. \quad (41)$$

where $n, m, k_i, i = 1 \dots 5$ are arbitrary constants such that $k_4 = k_5 \neq 0$ and $k_1 = k_2 = k_3 \neq 0$ are the conservation laws for the $N = 3$ generalized Riemann equation. Indeed we can easily verify it, using (17) showing, that

$$H_{n,m,t} = -(u_1 H_{n,m})_x. \quad (42)$$

We can construct the infinite number of the non-polynomial conservation laws using the analog of recursion operator as follows.

In the papers [4, 5] we proved that if some function H , which depends on u_n and its derivatives satisfy

$$H_t = -k u_{1,x} H - u_1 H_x. \quad (43)$$

then $\int dx H^{1/k}$ is a conserved Hamiltonian functionals for the generalized Riemann equation.

Moreover we have three additional lemmas

Lemma1: If f and g satisfy

$$f_t = -\mu u_{1,x} f - u_1 f_x, \quad g_t = -\nu u_{1,x} g - u_1 g_x. \quad (44)$$

and n, m are an arbitrary numbers such that $n\mu \pm m\nu \neq 0$ then

$H = \int (f^n g^m)^{1/(n\mu \pm m\nu)} dx$ is conserved

Lemma2: If f and g satisfy

$$f_t = -\mu u_{1,x} f - u_1 f_x, \quad g_t = -\mu u_{1,x} g - u_1 g_x, \quad (45)$$

then $H = \int (f \pm m)^{1/\mu} dx$ is conserved

Lemma3: If f satisfy

$$f_t = -\mu u_{1,x} f - u_1 f_x \quad (46)$$

then $g = u_{N,x}^m f$ satisfy $g_t = -(\mu + m)u_{1,x}g - u_1 g_x$ and $H = \int g^{\frac{1}{\mu+m}} dx$ is conserved.

The proofs of these lemmas are elementary.

Let us define the following operator

$$R_k = \frac{1}{u_{N,x}} \partial - k \frac{u_{N,xx}}{u_{N,x}^2} = u_{N,x}^{k-1} \partial \frac{1}{u_{N,x}^k}, \quad (47)$$

where k is an arbitrary number but $k \neq 0$.

We have the following

Theorem : If some function H which depends on u_n and its derivatives satisfy

$$H_t = -k u_{1,x} H - u_1 H_x \quad (48)$$

then $H_m = R_k^m H$ where $m = 0, 1, 2, \dots$ generates the infinite number of non-polynomial dispersive and dispersionless conserved Hamiltonian functionals $\int dx H_m^{1/k}$.

Proof: We carry it by induction. For $m = 0$ it is valid from the assumption because $H_0 = H$. Now we show that

$$(H_{m+1})_t = -k u_{1,x} H_{m+1} - u_1 H_{m+1,x}. \quad (49)$$

We have

$$(H_{m+1})_t = (R_k H_m)_t = R_{k,t} H_m - k R_k u_{1,x} H_m - R_k u_1 H_{m,x} = -k u_{1,x} R_k H_m - u_1 (R_k H_m)_x, \quad (50)$$

what can be rewritten as

$$\begin{aligned} R_{k,t} &= \left(\frac{1}{u_{N,x}} \right)_t \partial + k \left(\frac{1}{u_{N,x}} \right)_{x,t} = [R_k, k u_{1,x} + u_1 \partial] = \\ &= k \left(\frac{u_{1,x}}{u_{N,x}} - u_1 \left(\frac{1}{u_{N,x}} \right)_x \right) \partial + \left(\frac{u_{1,x}}{u_{N,x}} - u_1 \left(\frac{1}{u_{N,x}} \right)_x \right) \partial. \end{aligned} \quad (51)$$

and it is identically satisfied if we compute $R_{k,t}$ using the equation of motion.

Notice that for $k = 1$ we obtain trivial result because then RH is a total derivative.

For $N = 2$ we can use the functions Z_1, Z_2 defined in Esq.15 while for $N = 3$ the functions $P_i, i = 0, 1, 2, 3, S_i, i = 1, 2$ defined in Esq. 19 respectively as the seeds solutions in order to generates by R operator an infinite number of conserved Hamiltonian functionals.

Finally let us discuss the problem of the existence of conservation laws for generalized Riemann equation, which are in the form

$$H_t = -(u_1 H)_x. \quad (52)$$

Such conservation laws could be easily obtained using the Lemma1. Indeed

$$\int dx H_{k,n,m} = \int dx \frac{R_k^n G_k}{R_{k-1}^m F_{k-1}} \quad (53)$$

is conserved Hamiltonian functionals, where G_k, F_{k-1} are such that

$$G_{k,t} = -k u_{1,x} G_k - u_1 G_{k,x}, \quad F_{k-1,t} = -(k-1) u_{1,x} F_{k-1} - u_1 F_{k-1,x} \quad (54)$$

Example: only for the Hunter - Saxton equation for which $R_k = u_{1,x}^{2k-2} \partial \frac{1}{u_{1,x}^{2k}}$

$$\begin{aligned} G_3 &= u_{1,xx} u_{1,x}^2, \quad F_2 = u_{1,xx}, \\ H_{3,0,1} &= \frac{u_{1,xx} u_{1,x}^5}{u_{1,xxx} u_{1,x} - 4 u_{1,xx}^2}, \\ H_{3,2,0} &= \frac{u_{1,4x}}{u_{1,xx} u_{1,x}} - 14 \frac{u_{1,xxx}}{u_{1,x}^3} + 28 \frac{u_{1,xx}^2}{u_{1,x}^4}, \\ H_{3,2,1} &= \frac{u_{1,4x} u_{1,x}^2 - 14 u_{1,xxx} u_{1,xx} u_{1,x} + 28 u_{1,x}^3}{u_{1,xxx} u_{1,x}^2 - 4 u_{1,xx}^2 u_{1,x}}. \end{aligned} \quad (55)$$

$$\begin{aligned} G_4 &= u_{1,xx}^{\frac{3}{2}} u_{1,x}^2, \quad F_3 = u_{1,xx}^{\frac{3}{2}}, \\ H_{4,2,0} &= \frac{3 u_{1,4x}}{2 u_{1,xx} u_{1,x}^2} + \frac{3 u_{1,xxx}^2}{4 u_{1,xx}^2 u_{1,x}^2} - 27 \frac{u_{1,xxx}}{u_{1,x}^3} + 54 \frac{u_{1,xx}^2}{u_{1,x}^4}. \end{aligned} \quad (56)$$

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